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Exact Bending Solution of Inhomogeneous Plates from Homogeneous Thin-Plate Deflection

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Based on the first-order shear deformation theory, an exact solution for sandwich plates with dissimilar facings by the use of the potential function technique is presented. The solution is expressed in terms of the deflection of a homogeneous Kirchhoff thin plate and is valid for simply supported polygonal plates made of isotropic materials. Because of its mathematical similarity, the solution for functionally graded plates is also presented.

I. Introduction

SINCE Reissner[1] proposed a finite deflection theory for sandwich plates, research on sandwich structures has received great attention.[2–12] Noor et al.[11] recently presented a comprehensive survey of the status, achievements, and trends in the modeling of sandwich plates and shells. In particular, an interesting line of investigation is the linking relationships between the bending solutions of polygonal sandwich plates[2,3,12] and classical Kirchhoff thin plates. Because of its mathematical similarity,[7–9] the exact relationship regarding sandwich plates also applies to the first-order shear deformation theory for single-layer homogeneous plates.[13] The study was also extended to Reddy's third-order theory[14,15] for isotropic plates[16] and for symmetrically laminated plates composed of transversely isotropic laminae.[17] These exact connections apply to plates that are symmetric in the midplane. For plates in an asymmetric configuration, such as sandwich plates with dissimilar facing sheets and functionally graded plates,[18] however, a new study is needed. This paper addresses the exact solution for polygonal simply supported sandwich plates with dissimilar facings in terms of homogeneous thin-plate deflection. Such a relationship between functionally graded plates and classical plates is readily obtained because of mathematical similarity.

For orthotropic core materials in reality, two different values of shear moduli should be given.[4] The present technique for finding the linking relationships between inhomogeneous and homogeneous plates will not work in that case.

II. Field Equations

Consider an undeformed sandwich plate with dissimilar facing sheets in a Cartesian coordinate system $\{x_i\}$ $(i = 1, 2, 3)$, where the midplane of the sandwich core coincides with $x_3 = 0$. The facing sheets and core are composed of different isotropic elastic materials, with uniform thicknesses $h^{\pm}$ for the upper and lower facing sheets and h^c for the core.

In what follows, a subscript comma followed by an index i denotes a partial derivative with respect to the corresponding spatial coordinate x_i. The Einsteinian summation convention applies, unless otherwise specified, to repeated indices of tensor components, with Latin subscripts ranging from 1 to 3 and Greek subscripts from 1 to 2.

Under Reissner's assumptions,[1] the facing sheets are very thin $(h^{\pm} \ll h^c)$ such that they behave as two membranes, and the core undergoes only transverse shear deformation. The displacement field is then assumed in the form

$$v_\alpha^+(x_i) = u_\alpha + \tfrac{1}{2}(h^c + h^+)\psi_\alpha, \qquad v_3^+(x_i) = u_3$$

$$v_\alpha^-(x_i) = u_\alpha - \tfrac{1}{2}(h^c + h^-)\psi_\alpha, \qquad v_3^-(x_i) = u_3$$

$$v_\alpha^c(x_i) = u_\alpha + x_3\psi_\alpha, \qquad v_3^c(x_i) = u_3 \tag{1}$$

where u_α, u_3, and ψ_α are independent of x_3. For the bending problem of a sandwich plate under an arbitrarily distributed normal load $q(x_\alpha)$, the field equations are expressed as

$$N_{\alpha\beta,\beta} = 0, \qquad Q_{\alpha,\alpha} + q = 0, \qquad M_{\alpha\beta,\beta} - Q_\alpha = 0 \tag{2}$$

where

$$\begin{pmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} A_1 - B_1 & A_2 - B_2 \\ A_2 - B_2 & A_3 - B_3 \end{pmatrix} \begin{pmatrix} u_{\omega,\omega} \\ \psi_{\omega,\omega} \end{pmatrix} \delta_{\alpha\beta} + \frac{1}{2} \begin{pmatrix} B_1 & B_2 \\ B_2 & B_3 \end{pmatrix} \begin{pmatrix} u_{\alpha,\beta} + u_{\beta,\alpha} \\ \psi_{\alpha,\beta} + \psi_{\beta,\alpha} \end{pmatrix}$$

$$Q_\alpha = C(u_{3,\alpha} + \psi_\alpha) \tag{3}$$

In these expressions,

$$A_1 = \frac{E^+h^+}{1 - (\nu^+)^2} + \frac{E^-h^-}{1 - (\nu^-)^2}$$

$$A_2 = \frac{E^+h^+(h^c + h^+)}{2[1 - (\nu^+)^2]} - \frac{E^-h^-(h^c + h^-)}{2[1 - (\nu^-)^2]}$$

$$A_3 = \frac{E^+h^+(h^c + h^+)^2}{4[1 - (\nu^+)^2]} + \frac{E^-h^-(h^c + h^-)^2}{4[1 - (\nu^-)^2]}$$

$$B_1 = \frac{E^+h^+}{1 + \nu^+} + \frac{E^-h^-}{1 + \nu^-}$$

$$B_2 = \frac{E^+h^+(h^c + h^+)}{2(1 + \nu^+)} - \frac{E^-h^-(h^c + h^-)}{2(1 + \nu^-)}$$

$$B_3 = \frac{E^+h^+(h^c + h^+)^2}{4(1 + \nu^+)} + \frac{E^-h^-(h^c + h^-)^2}{4(1 + \nu^-)}$$

$$C = G^c h^c \tag{4}$$

where $E^{\pm}$ and $\nu^{\pm}$ denote Young's moduli and Poisson's ratios for the facing materials, respectively, and G^c denotes the shear modulus

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of the core. Note that A_2 and B_2 will vanish for a sandwich plate symmetric in its midplane.

With the expressions of Eqs. (3), field equations (2) are then expressed in terms of five displacement functions u_α, u_3, and ψ_α as

$$\tfrac{1}{2}B_1 u_{\alpha,\beta\beta} + \left(A_1 - \tfrac{1}{2}B_1\right)u_{\beta,\beta\alpha} + \tfrac{1}{2}B_2\psi_{\alpha,\beta\beta} + \left(A_2 - \tfrac{1}{2}B_2\right)\psi_{\beta,\beta\alpha} = 0 \tag{5}$$

$$C(u_{3,\alpha\alpha} + \psi_{\alpha,\alpha}) + q = 0 \tag{6}$$

$$\tfrac{1}{2}B_2 u_{\alpha,\beta\beta} + \left(A_2 - \tfrac{1}{2}B_2\right)u_{\beta,\beta\alpha} + \tfrac{1}{2}B_3\psi_{\alpha,\beta\beta} + \left(A_3 - \tfrac{1}{2}B_3\right)\psi_{\beta,\beta\alpha} - C(u_{3,\alpha} + \psi_\alpha) = 0 \tag{7}$$

Equations (6) and (7) may be recast, after a little manipulation with the help of Eq. (5), into the following form:

$$\left(A_1A_3 - A_2^2\right)\psi_{\alpha,\alpha\beta\beta} + A_1 q = 0 \tag{8}$$

$$(A_2B_1 - A_1B_2)u_{\beta,\beta\alpha} + \tfrac{1}{2}\left(B_1B_3 - B_2^2\right)\psi_{\alpha,\beta\beta} + \left(A_3B_1 - A_2B_2 + \tfrac{1}{2}B_2^2 - \tfrac{1}{2}B_1B_3\right)\psi_{\beta,\beta\alpha} - B_1C(u_{3,\alpha} + \psi_\alpha) = 0 \tag{9}$$

Following the technique[2,3] proposed for the analysis of symmetrical sandwich plate problems, two new functions, w and f, are introduced such that

$$\psi_\alpha = w_{,\alpha} + \varepsilon_{\alpha\omega} f_{,\omega} \tag{10}$$

where $\varepsilon_{\alpha\omega}$ is the two-dimensional permutation tensor. The expression of Eq. (10) is sufficient; however, w and f will not be uniquely determined. This is because the Cauchy–Riemann equations

$$w^*_{,\alpha} + \varepsilon_{\alpha\omega} f^*_{,\omega} = 0 \tag{11}$$

always give a solution $(f^* + iw^*)$ that is an analytic function of the complex variable $(x_1 + ix_2)$. Expression (10) remains unchanged when w and f are changed simultaneously by increments w^* and f^*.

Substituting Eq. (10) into Eqs. (5), (8), and (9) yields

$$\tfrac{1}{2}B_1 u_{\alpha,\beta\beta} + \left(A_1 - \tfrac{1}{2}B_1\right)u_{\beta,\beta\alpha} + A_2 w_{,\alpha\beta\beta} + \tfrac{1}{2}B_2\varepsilon_{\alpha\omega} f_{,\omega\beta\beta} = 0 \tag{12}$$

$$\left(A_1A_3 - A_2^2\right)w_{,\alpha\alpha\beta\beta} + A_1 q = 0 \tag{13}$$

$$[(A_2B_1 - A_1B_2)u_{\beta,\beta} + (A_3B_1 - A_2B_2)w_{,\beta\beta} - B_1C(u_3 + w)]_{,\alpha} + \varepsilon_{\alpha\omega}\left[\tfrac{1}{2}\left(B_1B_3 - B_2^2\right)f_{,\beta\beta} - B_1Cf\right]_{,\omega} = 0 \tag{14}$$

Equation (14) is a Cauchy–Riemann equation, which can be interpreted in an alternative form:

$$\tfrac{1}{2}\left(B_1B_3 - B_2^2\right)f_{,\beta\beta} - B_1Cf + i[(A_2B_1 - A_1B_2)u_{\beta,\beta} + (A_3B_1 - A_2B_2)w_{,\beta\beta} - B_1C(u_3 + w)] = F(x_1 + ix_2) \tag{15}$$

where $F(x_1 + ix_2)$ is an analytic function. Furthermore, when Eq. (15) is viewed as a nonhomogeneous partial differential equation with respect to unknowns (u_α, u_3, w, f), its solution is composed of the sum of a homogeneous general solution and an arbitrary particular solution. Because both the real and imaginary parts of function $F(x_1 + ix_2)$ are harmonic functions, the particular solution $(u^*_\alpha, u^*_3, w^*, f^*)$ can be taken as

$$u^*_\alpha = 0 \tag{16a}$$

$$u^*_3 = 0 \tag{16b}$$

$$-B_1C(f^* + iw^*) = F(x_1 + ix_2) \tag{16c}$$

and therefore Eq. (16c) fulfills Cauchy–Riemann condition (11). As a result of Eqs. (16a), (16b), and (11), the particular solution has a trivial contribution to u_α, u_3, and ψ_α, and thus to displacements and stresses of sandwich plates. Consequently, the particular solution of Eq. (15) can be discarded. Only the corresponding homogeneous part of Eq. (15) is of interest; that is,

$$\tfrac{1}{2}\left(B_1B_3 - B_2^2\right)f_{,\beta\beta} - B_1Cf = 0 \tag{17}$$

$$(A_2B_1 - A_1B_2)u_{\beta,\beta} + (A_3B_1 - A_2B_2)w_{,\beta\beta} - B_1C(u_3 + w) = 0 \tag{18}$$

Note that the unknown f has been decoupled with four other unknowns u_α, u_3, and w in field equations (13), (17), and (18), but coupled in Eq. (12). After simplification this new form of field equations is useful in some applications, such as in problems seeking fundamental solutions. Here it should be indicated that the unknown f is also coupled with u_α, u_3, and w through boundary conditions in the majority of problems. For simply supported edges of polygonal plates, however, the unknown f can be separately determined.

III. Simply Supported Rectilinear Edge

With the assumption that a sandwich plate is simply supported on its boundary Γ, the boundary condition is expressed as

$$N_{NN} = 0 \tag{19a}$$

$$M_{NN} = 0 \tag{19b}$$

$$u_3 = 0 \tag{19c}$$

$$u_T = 0 \tag{19d}$$

$$\psi_T = 0 \tag{19e}$$

where the uppercase subscripts N and T denote normal and tangential directions, respectively, on boundary Γ. No implicit summation applies to the repeated uppercase subscripts. For a polygonal plate, Eqs. (19a) and (19b) reduce to

$$u_{N,N} = 0, \qquad \psi_{N,N} = 0 \tag{20}$$

With the aid of Eqs. (10) and (18), Eqs. (19c–19e) and (20) can be further rewritten as

$$u_T = 0 \tag{21a}$$

$$u_{N,N} = 0 \tag{21b}$$

$$w = 0 \tag{21c}$$

$$w_{,NN} = 0 \tag{21d}$$

$$f_{,N} = 0 \tag{21e}$$

It is shown by Eqs. (21) that the unknowns u_α, u_3, w, and f have been decoupled in the simply supported boundary condition.

When the Gaussian theorem is applied in the plate plane Ω for the function $f_{,\alpha}$, that is,

$$\iint_\Omega f_{,\alpha\alpha}\,\mathrm{d}\Omega = \oint_\Gamma f_{,\alpha}n_\alpha\,\mathrm{d}\Gamma = \oint_\Gamma f_{,N}\,\mathrm{d}\Gamma \tag{22}$$

it can be concluded, through Eqs. (17) and (21e), that

$$f \equiv 0 \quad \text{on} \quad \Omega \tag{23}$$

This means that, for bending problems of simply supported polygonal sandwich plates, only four functions u_α, u_3, and w have to be solved. The field equations for u_α and w are Eqs. (12) and (13), consistently associated with Eqs. (21a–21d). The deflection u_3 can be simply obtained from Eq. (18) after u_α and w are solved.

IV. Solution Relationship

The field equation for a classical Kirchhoff homogeneous thin plate with its flexural stiffness D is

$$-Du^K_{3,\alpha\alpha\beta\beta} + q = 0 \tag{24}$$

The boundary condition of simply supported edges is

$$u^K_3 = 0, \qquad u^K_{3,NN} = 0 \tag{25}$$

From the uniqueness theorem of solution, it can be concluded from the analogy between field equations (13) and (24) and between boundary conditions (21c), (21d), and (25) that

$$\left(A_3 - A_2^2 / A_1\right)w = -Du^K_3 \tag{26}$$

Because $f = 0$, Eq. (12) associated with the boundary condition of Eqs. (21a–21d) has the following solution:

$$u_\alpha = -(A_2/A_1)w_{,\alpha} \tag{27}$$

Therefore, u_α and u_3 of the sandwich plate can be explicitly obtained from Eqs (18), (26), and (27) as

$$u_\alpha = \left[A_2 / \left(A_1A_3 - A_2^2\right)\right]Du^K_{3,\alpha}$$

$$u_3 = \left[A_1 / \left(A_1A_3 - A_2^2\right)\right]Du^K_3 - (1/C)Du^K_{3,\alpha\alpha} \tag{28}$$

which are expressed in terms of the classical thin-plate deflection. These are exact explicit relationships between the solution for simply supported polygonal sandwich plates and the deflection of Kirchhoff homogeneous thin plates. Once the Kirchhoff deflection solution u^K_3 is given, the solution using the asymmetrical sandwich plate theory is readily obtained through a simple manipulation of differentiation. This is easily achieved, as there are plenty of available solutions for classical homogeneous plates.

For the special case of symmetrical sandwich plates, $A_2 = 0$, Hu[2] gave the relationship between deflections of Reissner sandwich plates and classical Kirchhoff thin plates; also see the linear counterpart in the nonlinear work[6] for sandwich plates. Equations (28) represent a generalization of the connection[2] concerning symmetrical sandwich plates to the sandwich plates with dissimilar facings.

V. Functionally Graded Plates

The term "functionally graded material" refers to materials whose material properties vary along the thickness direction. The mixture of two distinct material phases such as ceramic and metal phases is a typical example of such materials. In the case of functionally graded plates composed of isotropic gradient materials,[18] the material moduli are, with respect to x_3 only,

$$E \equiv E(x_3), \qquad \nu \equiv \nu(x_3) \tag{29}$$

where E and ν denote Young's modulus and Poisson's ratio, respectively.

Based on the first-order shear deformation theory, the derivation and conclusion in the foregoing sections for sandwich plates are the same for the functionally graded plates except that displacement field (1) is replaced by

$$v_\alpha(x_i) = u_\alpha + x_3\psi_\alpha, \qquad v_3(x_i) = u_3 \tag{30}$$

and the coefficient expressions in Eqs. (4) are replaced by

$$[A_1, A_2, A_3] = \int_{-h/2}^{h/2}\left[1, x_3, x_3^2\right]\frac{E}{1-\nu^2}\,dx_3$$

$$[B_1, B_2, B_3] = \int_{-h/2}^{h/2}\left[1, x_3, x_3^2\right]\frac{E}{1+\nu}\,dx_3$$

$$C = \kappa\int_{-h/2}^{h/2} G\,dx_3 \tag{31}$$

where h is the plate thickness, G is the shear modulus, and κ is the shear correction factor.

VI. Conclusions

An exact solution for a sandwich plate with dissimilar facings and for a functionally graded plate has been found in terms of the deflection of a homogeneous Kirchhoff thin plate. The solution is valid for plates with simply supported rectilinear edges and that made of isotropic materials. The first-order shear deformation plate theory has been employed. It should be noted that the potential function technique used herein does not work for a plate made of an orthotropic material.

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